into the expressions for $\lambda_{\nu,\varkappa}^{+}$ and $h_{\nu,\varkappa}^{+}$ from (1.19) and (1.21).

We note that the results obtained here can be applied to a fairly large number of contact and mixed problems of elasticity theory as well as to modified mixed problems of mathematical physics. The need to tabulate the functions $S_{\nu}^{\varkappa(3)}(x, \theta)$ $(0 \leqslant x < \infty)$ and $G_{\nu,\varkappa}^{\pm}(\arccos x)$ $(|x| < 1)$ arises here; this can be achieved by using continued fractions /10-12/.

REFERENCES

1. RAKOV A. KH. and RVACHEV V.L., A contact problem of elasticity theory for a half-space whose elastic modulus is a power-law function of the depth, Dokl. Akad. Nauk UkrSSR, Ser. A, No.3, 1961.
2. TITCHMARSH E.C., Eigenfunction Expansions Associated with Second-Order Differential Equations /Russian translation/, Vol. 1, Izdat. Inostr. Lit, Moscow, 1960.
3. ARUTIUNIAN N. KH., A plane contact problem of creep theory, PMM, Vol.23, No.5, 1959.
4. POPOV G. YA., Contact Problems for a Linearly Deformable Base, Vishcha Shkola, Kiev-Odessa, 1982.
5. POPOV G. YA., Elastic Stress Concentration around Stamps, Slits, Thin Inclusions, and Reinforcements, Nauka, Moscow, 1982.
6. BELWARD J.A., On the relationship of some Fredholm integral equations of the first kind to a family of boundary value problems, J. Math. Analysis and Appl. Vol.48, No.1, 1974.
7. ALEKSANDROV V.M., KOVALENKO E.V. and MKHITARYAN S.M., On a method of obtaining spectral relationships for integral operators of mixed problems of the mechanics of continuous media, PMM, Vol.46, No.6, 1982.
8. MKHITARYAN S.M., On two spectral relationships for integral operators in a semi-infinite interval and their application to mixed problems, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.1, 1983.
9. MKHITARYAN S.M., On certain spectral relationships associated with the Carlemann integral equation and their applications to contact problems, PMM, Vol.47, No.2, 1983.
10. BATEMAN H. and ERDELYI A., Higher Transcendental Functions /Russian translation/, Nauka, Moscow, 1967.
11. MEIXNER J., Klassifikation, Bezeichnung und Eigenschaften der Sphäroidfunktionen, Math. Nachr., Vol.5, 1951.
12. MEIXNER J. and SCHAFKE F.W., Mathieusche Funktionen und Sphäroidfunktionen mit Anwendungen auf physikalische und technische Probleme, Springer, Berlin-Heidelberg, 1954.
13. BATEMAN H. and ERDELYI A., Higher Transcendental Functions /Russian translation/, Vol.1, Nauka, Moscow, 1973.
14. MAGNUS W., OBERHETTINGER F. and SONI R.P., Formulas and Theorems for the Special Functions of Mathematical Physics, Springer, Berlin-Heidelberg-New York, 1966.

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ASYMPTOTIC SOLUTION OF THREE-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY OF EXTENDED PLANE SEPARATION CRACKS*

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A solution of three-dimensional elasticity theory problems for separation cracks occupying a plane domain with one characteristic dimension much smaller than the other is constructed by the method of matched asymptotic expansions (cracks that are extended along a certain plane curve). The appropriate terms of the expansion of the solution in a small parameter characterizing the extent of the crack are constructed using an integro-differential equation in the displacement of points of the crack surface. For cracks that are extended along a line, the representation of the integro-differential equation in terms of a two-dimensional Fourier transform is used, which substantially simplifies the calculation. In the general case, the expansion is executed directly in the equation written in x-space. The asymptotic expansion constructed is valid in the middle part of the

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crack, outside of certain small neighbourhoods of the ends of the curve along which the crack is drawn. The accuracy of the solution obtained is analysed, and formulas are presented for the crack aperture and the distribution of the stress intensity factors for specific kinds of cracks: cracks of elliptical planform, a ring and ring sector, crescents, bounded by parabolic arcs, narrow crescent domains, extended along a parabolic arc, etc. A comparison of the results with existing solutions of elliptical and annular cracks, as well as with numerical solutions constructed specially by a variational-difference method for cracks of different shape demonstrates the high efficiency of the asymptotic formulas obtained.

1. Rectilinear extended crack. We consider a homogeneous isotropic medium with a crack occupying the domain G in the $z=0$ plane. Oppositely directed normal forces

$$\sigma_{zz}^{+}(x,y,0)=\sigma_{zz}^{-}(x,y,0)=-p(x,y)\leqslant 0$$
$$\sigma_{xz}(x,y,0)=\sigma_{yz}(x,y,0)=0,\ (x,y)\in G$$

are applied to the crack surfaces (the superscripts plus and minus correspond to the upper and lower crack edges). There is no load at infinity. Then /1-3/, the tangential components of the displacement of the crack surfaces are continuous

$$u_x^{+}(x,y,0)=u_x^{-}(x,y,0),\ u_y^{+}(x,y,0)=u_y^{-}(x,y,0),\ (x,y)\in G$$

and we have for the normal components of the displacement

$$u_z^{+}(x,y,0)=-u_z^{-}(x,y,0)=u(x,y)\geqslant 0,\ (x,y)\in G$$

Determination of the displacement of separation crack surfaces reduces to seeking a bounded function $u(x,y)$ that equals zero outside the domain G and satisfies the following integro-differential equation for $(x,y)\in G$

$$\Delta_{xy}\iint\limits_G \frac{u(x',y')}{r}\,dx'\,dy'=-2\pi\beta p(x,y),\quad (x,y)\in G \tag{1.1}$$
$$r^2=(x-x')^2+(y-y')^2,\ \beta=\mu(1-\nu)^{-1}$$

Here μ and ν are, respectively, the shear modulus and Poisson's ratio of the medium, and Δ_{xy} is the two-dimensional Laplace operator.

Equation (1.1) can be written in the form

$$P_G\{F_{xy}^{-1}[\sqrt{\xi_x^2+\xi_y^2}]*u(x,y)\}=\beta p(x,y),\quad (x,y)\in G \tag{1.2}$$

where F_{xy} is the Fourier transform

$$F_{xy}[\varphi(x,y)]=\iint\limits_{-\infty}^{\infty}\exp[i(\xi_x x+\xi_y y)]\,\varphi(x,y)\,dx\,dy$$

P_G is the operator of the constraint on the domain G and the functions in (1.2) are understood in the generalized sense $(u\in S'(R^2),\ \sqrt{\xi_x^2+\xi_y^2}\in S'(R^2),\ p\in S'(G),\ F_{xy}:S'(R^2)\to S'(R^2),\ P_G:S'(R^2)\to S'(G)$, [4]).

Let the crack occupy a domain $G(\varepsilon)$ of the following form (Fig. 1): $|x|\leqslant L,\ |y|\leqslant\varepsilon\rho(x)$, where $L>0$, the function $\rho(x)$ is bounded and $\rho(x)\in C^2(-L,L)$, $\rho\geqslant 0$ and the dimensionless parameter $\varepsilon>0$. For small ε we obtain a narrow crack stretched along the Ox axis. The problem is to determine the asymptotic of the edge displacements $u(x,y,\varepsilon)$ (corresponding to the crack $G(\varepsilon)$) as $\varepsilon\to 0$.

We introduce the internal coordinate $Y=\varepsilon^{-1}y$. Then the crack will occupy a constant domain G: $|x|\leqslant L,\ |Y|\leqslant\rho(x)$ in the x, Y coordinates. In the Fourier transforms $\xi_Y=\varepsilon\xi_y$ corresponds to the coordinate Y; hence

$$F_{xy}^{-1}[\sqrt{\xi_x^2+\xi_y^2}]=\varepsilon^{-1}F^{-1}[\sqrt{\xi_x^2+\xi_Y^2\varepsilon^{-2}}],\quad F=F_{xY} \tag{1.3}$$

Substituting (1.3) into (1.2) and taking into account that the convolution is reduced ε times in the new (x,Y) coordinates, we obtain.

$$P_G\{F^{-1}[\sqrt{\varepsilon^2\xi_x^2+\xi_Y^2}]*u(x,Y,\varepsilon)\}=\varepsilon\beta p(x,Y,\varepsilon),\quad (x,Y)\in G \tag{1.4}$$

We find the asymptotic $F^{-1}[\sqrt{\varepsilon^2\xi_x^2+\xi_Y^2}]$ as a generalized functions as $\varepsilon\to 0$. The following asymptotic forms are derived directly from the definition of the generalized functions by standard methods of regularizing integrals:

$$(\varepsilon^2\xi_x^2+\xi_Y^2)^{-1/2} = \eta + o(1), \quad \eta = \eta(\xi_x,\xi_Y,\varepsilon) = P\frac{1}{|\xi_Y|} + \ln\frac{4}{\varepsilon^2\xi_x^2}\delta(\xi_Y)$$
$$\xi_Y^2(\varepsilon^2\xi_x^2+\xi_Y^2)^{-1/2} = |\xi_Y| + \frac{\varepsilon^2}{2}\xi_x^2\left[P\frac{1}{|\xi_Y|} + 2\delta(\xi_Y) - 2\eta\right] + o(\varepsilon^2)$$
$$\left(P\frac{1}{|\xi_Y|},\varphi\right) = \iint_{-\infty}^{\infty}[\varphi(\xi_x,\xi_Y) - \theta(1-|\xi_Y|)\varphi(\xi_x,0)]\frac{d\xi_x\, d\xi_Y}{|\xi_Y|}$$
$$\varphi(\xi_x,\xi_Y)\in S(R^2)$$

We hence have

$$\sqrt{\varepsilon^2\xi_x^2+\xi_Y^2} = (\varepsilon^2\xi_x^2+\xi_Y^2)(\varepsilon^2\xi_x^2+\xi_Y^2)^{-1/2} = |\xi_Y| + {}^1\!/_2\varepsilon^2\xi_x^2[\delta(\xi_Y)+\eta] + o(\varepsilon^2) \qquad (1.5)$$

Since /4/

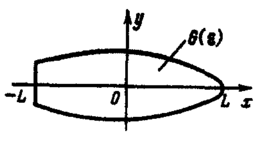


Fig. 1

$$F^{-1}[|\xi_Y|] = \frac{\delta(x)}{\pi}\frac{\partial}{\partial Y}P\frac{1}{|Y|}, \quad F^{-1}[\xi_x^2\delta(\xi_Y)] = -\frac{\delta''(x)}{2\pi}$$
$$F^{-1}\left[\xi_x^2 P\frac{1}{|\xi_Y|}\right] = \frac{\delta''(x)}{\pi}(\gamma + \ln|Y|)$$
$$F^{-1}[\xi_x^2\ln|\xi_x|\delta(\xi_Y)] = \frac{1}{2\pi}\left[\gamma\delta''(x) + {}^1\!/_2\frac{\partial^2}{\partial x^2}P\frac{1}{|x|}\right]$$

where γ is Euler's constant, then because of the continuity of the operator $F^{-1}: S'(R^2)\to S'(R^2)$, formula (1.4) can be written in the form

$$P_G\{[\Phi_0 + \varepsilon^2\ln(2/\varepsilon)\delta''(x) + \varepsilon^2\Phi_2 + o(\varepsilon^2)] * u(x,Y,\varepsilon) = -2\pi\beta\varepsilon p(x,Y,\varepsilon) \qquad (1.6)$$
$$\Phi_0 = -2\delta(x)\frac{\partial}{\partial Y}P\frac{1}{Y}$$
$$\Phi_2 = \frac{\delta''(x)}{2} - \delta''(x)\ln|Y| + {}^1\!/_2\frac{\partial^2}{\partial x^2}P\frac{1}{|x|}$$

Let $p(x, Y, \varepsilon)$ have the following asymptotic

$$p(x,Y,\varepsilon) = \sum_{i=0}^{2}\varepsilon^i p_i(x,Y) + o(\varepsilon^2), \quad p_0, p_1, p_2 \in C(G)$$

Then it is natural to seek the asymptotic form $u(x, Y, \varepsilon)$ in the following form that results from comparing the asymptotic expansions of the right and left sides in (1.6):

$$u(x,Y,\varepsilon) = \varepsilon\{u_0(x,Y) + \varepsilon u_1(x,Y) + \varepsilon^2\ln(2/\varepsilon)v(x,Y) + \varepsilon^2 u_2(x,Y) + w(x,Y,\varepsilon)\}$$

The third component in the braces is necessary to cancel the term of order $\varepsilon^3\ln\varepsilon$ generated by the corresponding logarithmic term in the asymptotic expansion of the kernel in (1.6).

The supports of all the functions here lie in G; $u_0(x, Y)$ is a regular bounded continuous function, the functions u_1 u_2, v are regular and continuous in any closed domain not containing the ends $G(x = \pm L)$; $w(x,Y,\varepsilon) = o(1)$ (not $o(\varepsilon^2)$ because of the possible boundary layers at the ends of G whose area tends to zero, in which the quantity $\varepsilon^{-1}u(x,Y,\varepsilon) - u_0(x,Y)$ is bounded) and $w(x,Y,\varepsilon) = o(\varepsilon^2)$ in any closed domain not containing the ends G.

We will find u_0, u_1, u_2, v in the middle part of G (in any of its closed subdomains not containing the ends). We note that because of the continuity of the convolution operation in $S'(R^2)$ as well as because the support of the function Φ_0 and $\delta''(x)$ is the line $x = 0$ in the middle part of the crack, we have

$$\Phi_0 * w(x,Y,\varepsilon) = o(\varepsilon^2), \quad \varepsilon^2\Phi_2 * w(x,Y,\varepsilon) = o(\varepsilon^2)$$
$$\varepsilon^2\ln(2/\varepsilon)\delta''(x) * w(x,Y,\varepsilon) = o(\varepsilon^2)$$

Consequently, equating terms of identical order in (1.6), we obtain in the middle part of the crack

$$\Phi_0 * u_0 = -2\pi\beta p_0, \quad \Phi_0 * u_1 = -2\pi\beta p_1, \quad \Phi_0 * u_2 = -2\pi\beta p_2 - \Phi_2 * u_0 \qquad (1.7)$$
$$\Phi_0 * v = -\delta''(x) * u_0 = -\frac{d^2}{dx^2}\int_{-\rho(x)}^{\rho(x)} u_0(x,Y)\,dY$$

Equations (1.7) can be considered as equalities of continuous functions. For any fixed $x \in (-L, L)$ each is (as should be expected) an equation of the plane problem of a rectilinear crack of normal separation (with a certain force distribution along the edges) and all the equations can be solved successively in quadratures /1,5/.

We present the explicit version of the asymptotic form for $p(x, Y, \varepsilon) = p = \text{const}$. In this case we obtain from (1.7)

$$p_1 = p_2 = 0, \ u_1 = 0, \ u_0 = q, \ v = qf''/4$$
$$u_2 = {}^1/_8 q\left\{2f'' + (2\ln 2) f'' - (f\ln f)'' + \frac{d^2}{\partial x^2} T_\theta f\right\} =$$
$${}^1/_8 q\left\{2(1+\ln 2) f'' + (f\Omega)'' + \frac{d^2}{dx^2} Tf\right\}$$
$$f = f(x) = \rho^2(x)\,\theta(1 - |x/L|)$$
$$q = q(x, Y) = \beta p\sqrt{\rho^2(x) - Y^2}, \quad \Omega = \Omega(x) = \ln\frac{L^2 - x^2}{f(x)}$$
$$Tf = \int_{-L}^{L} \frac{f(x') - f(x)}{|x' - x|}dx', \quad T_\theta f = \int_{-\infty}^{\infty}\frac{f(x') - \theta(1 - |x' - x|) f(x)}{|x' - x|}dx'$$

and finally

$$u(x, Y, \varepsilon) = \varepsilon q Q + o(\varepsilon^3) \tag{1.8}$$
$$Q = Q(x, Y, \varepsilon) = 1 + ({}^1/_8)\varepsilon^2 [2\ln(4/\varepsilon) f'' + 2f'' + (f\Omega)'' + d^2(Tf)/dx^2]$$

If the function $f''(x)$ is integrable in $[-L, L]$ and $f''(x) \in C^1(-L, L)$, and f and f' are bounded in $[-L, L]$, then (1.8) can be represented in a somewhat different form. In fact, for $x \in (-L, L)$

$$\frac{d^2}{dx^2} T_\theta f = \frac{d^2}{dx^2}\left[P\frac{1}{|x|} * f(x)\right] =$$
$$P\frac{1}{|x|} * [f'' + f'(-L)\,\delta(x+L) - f'(L)\,\delta(x-L) +$$
$$f(-L)\,\delta'(x+L) - f(L)\,\delta'(x-L)]$$

where f'' is the derivative understood in the ordinary (not generalized) sense. Hence, for $x \in (-L, L)$

$$P\frac{1}{|x|} * \frac{d^2 f}{dx^2} = -f(L)(L-x)^{-2} - f(-L)(L+x)^{-2} - \tag{1.9}$$
$$f'(L)(L-x)^{-1} + f'(-L)(L+x)^{-1} + T_\theta f''$$
$$Q = 1 + ({}^1/_8)\varepsilon^2 [2\ln(4/\varepsilon) f'' + 2f'' + f''\ln(L^2 - x^2) +$$
$$Tf'' - (f\ln f'') - f(L)(L-x)^{-2} - f(-L)(L+x)^{-2} +$$
$$f'(L)(L-x)^{-1} - f'(-L)(L+x)^{-1}]$$

We note that when using (1.9) for Q the single difficulty is evaluation of the integral Tf'' therein. In a number of cases (Sec. 3), this integral is found analytically, and in the general case numerically. It is here convenient to utilize the properties of the operator T established in /6/. For the stress intensity factors N at the crack edge at the points $(x, \pm\varepsilon\rho(x))$ (for $x \in (-L, L)$) we obtain from (1.8)

$$N = p\sqrt{\varepsilon\rho(x)/2}\,[1 + \varepsilon^2\rho'^2(x)]^{1/4}\,[Q + o(\varepsilon^2)]$$

2. Curved extended crack. We now consider the more general case of a crack stretched along a certain smooth curve, given in the plane $z = 0$, without selfreentrances $\mathbf{R}(l), l \in [-L, L]$ (it can even be closed $\mathbf{R}(L) = \mathbf{R}(-L)$) of length $2L$, where l is its natural parameter (the distance along the curve from its midpoint along the length). Then

$$d\mathbf{R}(l)/dl = \boldsymbol{\tau}(l), \ \mathbf{n}(l) = \mathbf{e}_z \times \boldsymbol{\tau}(l) \tag{2.1}$$
$$d\boldsymbol{\tau}(l)/dl = -k(l)\,\mathbf{n}(l), \quad d\mathbf{n}(l)/dl = k(l)\,\boldsymbol{\tau}(l)$$

where $\boldsymbol{\tau}(l)$ and $\mathbf{n}(l)$ are the tangential and normal directions to the curve, $k(l)$ is its curvature at the point $\mathbf{R}(l)$ (positive or negative). We introduce the coordinates (l, m) in the $z = 0$ plane

$$\mathbf{x}(l, m) = \mathbf{R}(l) + \varepsilon m\rho(l)\,\mathbf{n}(l) \tag{2.2}$$

(the conditions on $\rho(l)$ are the same as in Sec. 1). Then the domain of the crack $G(\varepsilon)$ is given by the inequalities $|l| \leqslant L, |m| \leqslant 1$ (Fig. 2). As in Sec. 1, the problem is to determine the asymptotic of the displacement $u(l, m, \varepsilon)$ as $\varepsilon \to 0$. The Jacobian of the mapping given in

(2.2) equals

$$D(l, m) = |\partial(x, y) / \partial(l, m)| = \varepsilon\rho(l)[1 + \varepsilon m\rho(l) k(l)] \tag{2.3}$$

and we write (1.1) in the domain $G(\varepsilon)$ in the following form, taking (2.1), (2.2) and (2.3) into account:

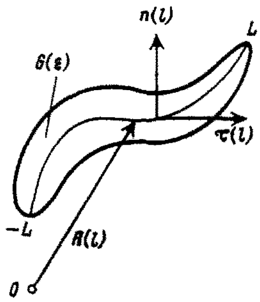


Fig.2

$$\Delta_{xy} \int_{-L}^{L} \int_{-1}^{1} \frac{u(l', m', \varepsilon)}{|\Delta \mathbf{x}|} D(l', m')\, dl'\, dm' = -2\pi\beta p(l, m, \varepsilon) \tag{2.4}$$

Here (compare /7/)

$$\begin{gathered}\Delta_{xy}\varphi(l, m) = \varkappa^{-2}\partial^2\varphi / \partial l^2 + [\varepsilon^{-2}\rho^{-2} + m^2\rho'^2\rho^{-2}\varkappa^{-2}]\, \partial^2\varphi / \partial m^2 - \\ 2m\rho'\rho^{-1}\varkappa^{-2}\partial^2\varphi / \partial l \partial m - \varepsilon m\rho k'\varkappa^{-3}\partial\varphi / \partial l + \\ [\varepsilon^{-1}\rho^{-1}\varkappa^{-1}k + m(2\rho'^2 - \rho\rho'')\rho^{-2}\varkappa^{-2} + \varepsilon m^2\rho' k'\varkappa^{-3}]\, \partial\varphi / \partial m \\ \varkappa = 1 + \varepsilon m\rho k, \quad \Delta\mathbf{x} = \mathbf{x}(l', m') - \mathbf{x}(l, m)\end{gathered} \tag{2.5}$$

As in Sec. 1, the asymptotic form of the operator

$$K_\varphi = \Delta_{xy} \int_{-L}^{L} \int_{-1}^{1} \frac{\varphi(l', m')}{|\Delta \mathbf{x}|} D(l', m')\, dl'\, dm'$$

must be found to obtain the asymptotic of the function $u(l, m, \varepsilon)$ from (2.4).

Since

$$\begin{gathered} K\varphi = \Delta_{xy}[\varepsilon H(\varphi\rho) + \varepsilon^2 H(\varphi m\rho^2 k)] \\ H\varphi = \int_{-L}^{L} \int_{-1}^{1} \frac{\varphi(l', m')}{|\Delta \mathbf{x}|}\, dl'\, dm' \end{gathered} \tag{2.6}$$

because of (2.3), it is sufficient to find the asymptotic form of the operator H and then to use relationship (2.6).

Let the function φ be fairly smooth, then

$$H\varphi = J(l, m) + \sum_{i=0}^{2} I_i(l, m), \quad J(l, m) = \int_{-L}^{L} \int_{-1}^{1} \frac{\psi}{|\Delta \mathbf{x}|}\, dl'\, dm' \tag{2.7}$$

Here

$$\begin{gathered} \psi = \psi(l', m', l, m) = \varphi(l', m') - \sum_{i=0}^{2} \frac{1}{i!} \frac{\partial^i \varphi}{\partial l^i}(l, m')(\Delta l)^i \\ \psi = O[(\Delta l)^3] \quad \text{as} \quad l' \to l \end{gathered} \tag{2.8}$$

$$\begin{gathered} I_i(l, m) = \int_{-1}^{1} \frac{1}{i!} \frac{\partial^i \varphi}{\partial l^i}(l, m') J_i(m', l, m)\, dm' \\ J_i(m', l, m) = \int_{-L}^{L} \frac{(\Delta l)^i}{|\Delta \mathbf{x}|}\, dl' \end{gathered} \tag{2.9}$$

According to (2.2)

$$\begin{gathered} |\Delta \mathbf{x}|^{-1} = A^{-1/2} - {}^1/_2 BA^{-3/2}\varepsilon - {}^1/_2 CA^{-3/2}\varepsilon^2 + {}^3/_8 B^2 A^{-5/2}\varepsilon^2 + o(\varepsilon^2) \\ A = A(l', l) = (\Delta \mathbf{R})^2 \\ B = B(l', m', l, m) = 2(\Delta \mathbf{R})\Delta(m\rho\mathbf{n}) \\ C = C(l', m', l, m) = [\Delta(m\rho\mathbf{n})]^2 \end{gathered} \tag{2.10}$$

Substituting (2.10) into the second formula in (2.7), we obtain the asymptotic form of the integral J. It is impossible to find the asymptotic form of the integrals J_i by using only the asymptotic form (2.10) for the integrands, since the terms in the expansion (2.10) become infinite for $l' = l$ and the corresponding integrals in (2.9) will diverge. To obtain the asymptotic forms of J_i it is necessary to replace the integrands by their composite asymptotic expansion /7-10/, which is the sum of the internal (for small $l' - l$) and external (obtained from (2.10) and corresponding to the original coordinates) asymptotic expansions diminished by the common part with the internal asymptotic expansion. As a result of calculations, taking (2.7) and (2.10) into account, we obtain the following asymptotic form of the operator H:

$$H\varphi = \int_{-L}^{L} [F(l')\,|\Delta \mathbf{R}|^{-1} - F(l)\,|\Delta l|^{-1}]\, dl' + F(l) \ln \frac{4g}{\varepsilon^2\rho^2} - \tag{2.11}$$
$$2\int_{-1}^{1} \varphi_0 \ln|\Delta m|\, dm' - \frac{\varepsilon}{2}\int_{-L}^{L}\int_{-1}^{1} \{B\,|\Delta \mathbf{R}|^{-3}\varphi_1 - \chi_1\,|\Delta l|^{-1}\varphi_0\} \times$$
$$dl'\,dm' + \varepsilon\int_{-1}^{1} \varphi_0\chi_1\Lambda_1\, dm' - \frac{\varepsilon^2}{2}\int_{-L}^{L}\int_{-1}^{1} \{C\,|\Delta \mathbf{R}|^{-3}\varphi_1 -$$
$$h_2\,|\Delta l|^{-3}(\varphi_1 - \psi) - h_1 f'\Delta l\,|\Delta l|^{-3}(\varphi_0 + \varphi'\Delta l) -$$
$$(m'^2\rho'^2 + m'm\chi^2 + h_1\rho\rho'' + {}^1/_8\chi_2)\,|\Delta l|^{-1}\varphi_0\}\, dl'\,dm' +$$
$$\frac{\varepsilon^2}{4}\int_{-1}^{1} \varphi' h_2 (1 - \Lambda)\, dm' + {}^3/_8\varepsilon^2 \int_{-L}^{L}\int_{-1}^{1} \{B^2\,|\Delta \mathbf{R}|^{-5}\varphi_1 - \chi_1^2\,|\Delta l|^{-1}\varphi_0\} \times$$
$$dl'\,dm' + \varepsilon^2\int_{-1}^{1} \varphi''(h_2 l g^{-1} + h_1 f'\Lambda_1)\, dm' +$$
$$\varepsilon^2\int_{-1}^{1} \varphi_0 \Big\{\frac{1}{2} h_2 (L^2 + l^2) g^{-2} + h_1 f' l g^{-1} + {}^1/_{48}\chi_2 (5 - 3\Lambda) +$$
$${}^1/_2 (m'^2\rho'^2 + m'm\chi^2 + h_1\rho\rho'')\Lambda_1 + \frac{3}{8}\chi_1^2\Lambda - \chi_1^2\Big\}\, dm' + o(\varepsilon^2)$$

Here

$$F(l) = \int_{-1}^{1} \varphi(l, m)\, dm,\ \ \chi = \rho(l)\, k(l),\ \ \chi_1 = (m' + m)\,\chi,\ \ \chi_2 =$$
$$(\Delta m)^2\chi^2,\quad g = L^2 - l^2,\ \ \Lambda = \ln[4g / (\varepsilon^2 h_2)],\ \ \Lambda_1 = 1 - \Lambda / 2,\ h_1 =$$
$$m'(\Delta m),\ h_2 = (\Delta m)^2\rho^2,\ \varphi_0 = \varphi(l, m'),\ \varphi_1 = \varphi(l', m'),\ \varphi' = \partial\varphi(l, m')/dl$$
$$\rho,\ \rho',\ \rho'',\ f,\ f',\ f''$$

are taken from the argument l.

Making the same assumptions about the form of the asymptotic forms $p(l, m, \varepsilon)$ and $u(l, m, \varepsilon)$ as in Sec. 1, by taking account of (2.12),(2.6),(2.5) and (2.4), we can obtain integro-differential equations for $u_0(l, m)$, $u_1(l, m)$, $u_2(l, m)$, $v(l, m)$, that generalize (1.7). To do this it is sufficient to show that the contribution of the remainder term $\varepsilon w(l, m, \varepsilon)$ containing the boundary layer at the ends has a lower order on the left side of (2.4) than the contribution of the other terms of the asymptotic form $u(l,m,\varepsilon)$. But this follows from the fact that the boundary layer contribution $u_k(x, y, \varepsilon)$ at the ends of $G(\varepsilon)$ in the left side of (1.1) equals

$$\Delta_{xy} \iint_{\delta S} \frac{u_k(x', y', \varepsilon)}{r}\, dx'\,dy' = o(\varepsilon^2)$$

where δS is the domain of boundary layer action with dimension $o(\varepsilon)$ (the longitudinal dimension $o(1)$, the transverse $O(\varepsilon)$), and (x, y) lies in the middle part of the crack and does not belong to δS, $u_k(x, y, \varepsilon) = O(\varepsilon)$. The equations for u_0, u_1, u_2, v reduce to the form

$$\partial^2 P(u_0) / \partial M^2 = \pi\beta p_0,\ 2\partial^2 P(u_1) / \partial M^2 = 2\pi\beta p_1 - k\partial P(u_0) / \partial M \tag{2.12}$$
$$2\partial^2 P(v \ln(2/\varepsilon) + u_2)/\partial M^2 = 2\pi\beta p_2 - k\partial P(u_1)/\partial M + I(U_0) +$$
$${}^1/_2\partial^2 [U_0 \ln(4g\,\varepsilon^{-2})]/\partial l^2 - \partial^2 P(u_0)/\partial l^2 - \partial^2 U_0/\partial l^2 +$$
$${}^1/_8 k^2 U_0 \ln(4g\varepsilon^{-2}) + k^2 M\partial P(u_0)/\partial M - {}^1/_4 k^2 P(u_0) + {}^7/_{12} k^2 U_0$$
$$M = \rho(l)\, m,\quad U_0(l) = \int_{-\rho(l)}^{\rho(l)} u_0(l, M)\, dM$$
$$I(U_0) = \int_{-L}^{L} \Big\{\frac{U_0(l')}{|\Delta \mathbf{R}|^3} - \frac{1}{|\Delta l|^3}\sum_{i=0}^{2} \frac{1}{i!}\frac{\partial^i U_0}{\partial l^i}(l)\,(\Delta l)^i -$$
$$\frac{1}{8|\Delta l|} k^2(l)\, U_0(l)\Big\}\, dl',\quad P(\varphi) = \int_{-\rho(l)}^{\rho(l)} \varphi(l, M') \ln|M - M'|\, dM'$$

For $p(l, m, \varepsilon) = p = \text{const}$, we obtain from (2.12)

$$u(l, m, \varepsilon) = \varepsilon\beta p\rho(l)\sqrt{1 - m^2}\, Q(l, m, \varepsilon) + o(\varepsilon^3) \tag{2.13}$$
$$Q(l, m, \varepsilon) = 1 - \frac{\varepsilon}{4}\chi m + \frac{\varepsilon^2}{4}\Big(f'' + \frac{\chi^2}{4}\Big)\ln\frac{4}{\varepsilon} + \frac{\varepsilon^2}{8}\frac{d^2(f\Omega)}{dl^2} +$$

$$\frac{\varepsilon^2}{32}\chi^2\Omega - \frac{\varepsilon^2}{8}f'' - \frac{\varepsilon^2}{12}\chi^2 + {}^3/_{16}\varepsilon^2\chi^2 m^2 + \frac{\varepsilon^2}{4}I(f)$$

and the stress intensity factor at the points $(l, \pm\varepsilon\rho(l))$ equals

$$N_\pm = p\sqrt{\varepsilon\rho(l)/2}\,[(1\pm\varepsilon\chi)^2+\varepsilon^2\rho'^2]^{1/4}(1\pm\varepsilon\chi)^{-3/4}\times[Q(l,\pm 1,\varepsilon)+o(\varepsilon^2)] \quad (2.14)$$

3. Examples. We present the explicit form of the asymptotic forms obtained from Secs. 1 and 2 for cracks of different shape for $p(x, y, \varepsilon) = p = \mathrm{const}$, and we compare the results with exact or numerical solutions.

1°. An elliptical crack

$$\rho(x) = \sqrt{L^2 - x^2},\quad Q = 1 - {}^1/_2\varepsilon^2(\ln(4/\varepsilon) - {}^1/_2)$$

The exact solution has the form ([1]): $u(x, Y, \varepsilon) = \varepsilon q / E(\sqrt{1-\varepsilon^2})$ and the first terms of its asymptotic being considered agree with those obtained. Graphs of $Q(\varepsilon)$ (curve *1*) and $1/E(\sqrt{1-\varepsilon^2})$ (curve *2*) are shown in Fig. 3. The accuracy of the asymptotic formulas (1.8) and (1.10) is of the order of 3% even for $\varepsilon = 0.5$, and is fractions of a percent for $\varepsilon < 0.25$.

2°. A crack bounded by parabolic arcs:

$$\rho(x) = L(1 - x^2/L^2)$$
$$Q = 1 + ({}^1/_8)\{(3\lambda^2 - 1)\ln[16\varepsilon^{-2}(1-\lambda^2)^{-1}] - 10\lambda^2 + 2\}\varepsilon^2$$
$$\lambda = x/L$$

The change in the intensity factor along the crack boundary is shown in Figs. 4a.and b (s is the distance from the middle along the boundary) for $p = 1$, $L = \varepsilon^{-1} = 4$ (Fig. 4a), and $L = \varepsilon^{-1} = 6$ (Fig. 4b; curve *1* is obtained by using the asymptotic formula (1.10) and curve *2* by using the formula $N_0 = p\sqrt{\varepsilon\rho/2}\,[1+\varepsilon^2\rho'^2]^{1/4}$ (corresponding to the plane problem approximation), and curve *3* numerically*. The discrepancy does not exceed 1 - 3% near the middle part of the crack.

3°. Crack in the shape of a generalized ellipse: $\rho(x) = L(1 - x^2/L^2)^{\zeta/2}$, $\zeta > 0$. For $x = 0$ (on the axis of crack symmetry)

$$Q = 1 - {}^1/_4\varepsilon^2[\ln(16\varepsilon^{-2}) - 1 - \Gamma'(\zeta)/\Gamma(\zeta) - \gamma]$$

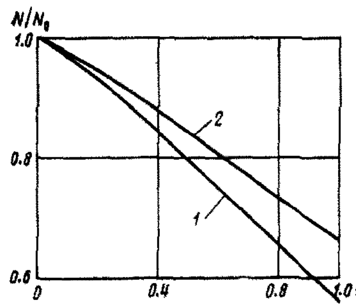


Fig.3

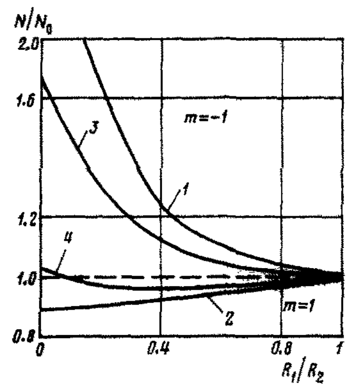
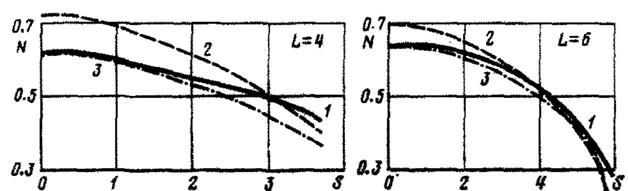


Fig. 5



a b

Fig.4

*The numerical solutions of the crack problems used for comparison in Examples 2-6 are constructed in the paper by Gol'dshtein, R.V., Otroshchenko, I.V., and Fedorenko, R.P., Method of refining boundary meshes in three-dimensional crack theory problems. Preprint, Institute of Mechanics Problems, USSR Academy of Sciences, No. 239, Moscow, 1984.

Given in the table are numerical (in parentheses) and asymptotic values of the intensity factor for $\zeta = 1/2$ and $\zeta = 3/2$ (referred to the intensity factor for the plane problem No).

3°		$\varepsilon = 1/4$	$\varepsilon = 1/8$
	$\zeta = 1/2$	0.954 (0.948)	0.977 (0.969)
	$\zeta = 3/2$	0.908 (0.905)	0.950 (0.948)
4°		$\varepsilon = 1/4$	$\varepsilon = 1/8$
	$\alpha_0 = \pi/3$, $m = 1$	0.941 (0.919)	0.970 (0.926)
	$m = -1$	1.066 (1.018)	1.032 (0.976)
	$\alpha_0 = \pi/2$, $m = 1$	0.950 (0.919)	0.973 (0.919)
	$m = -1$	1.076 (1.047)	1.035 (0.990)
5°		$\varepsilon = 1/3$	$\varepsilon = 1/6$
	$m = 1$	0.882 (0.643)	0.946 (0.919)
	$m = -1$	1.049 (0.863)	1.030 (1.032)

4°. Crack in the shape of an annular sector. The line $\mathbf{R}(l)$ has the shape of an arc of a circle of radius R of length $2L = 2\alpha_0 R$, $0 < \alpha_0 < \pi$, $\rho(l) = R$, $\chi = 1$. In this case

$$Q = 1 - {}^1/_4\varepsilon m + \frac{\varepsilon^2}{32}\left[\ln\left(256\varepsilon^{-2}\operatorname{tg}\frac{\alpha_1}{2}\operatorname{tg}\frac{\alpha_2}{2}\right) - \cos\alpha_1\operatorname{ctg}\alpha_1 - \cos\alpha_2\operatorname{ctg}\alpha_2 - 3 + 6m^2\right],\quad \alpha = \frac{l}{R},\quad \alpha_1 = \frac{\alpha_0 - \alpha}{2},\quad \alpha_2 = \frac{\alpha_0 + \alpha}{2}$$

In the case of an annular crack ($\alpha_0 = \pi$, inner radius $R_1 = (1 - \varepsilon)R$, and outer radius $R_2 = (1 + \varepsilon)R$)

$$Q = 1 - 0{,}25\varepsilon m + \varepsilon^2[\ln(256\varepsilon^{-2}) - 3 + 6m^2]/32 \tag{3.1}$$

and the asymptotic form (2.14) agrees with /11/. Given in Fig. 5 is the dependence of $N_\pm/N_0$ on the ratio R_1/R_2 determined numerically (by using data in /12/, curves *1* and *2*), and by the asymptotic formulas (curves *3* and *4*). For $R_1/R_2 = 0.5$, the error is 1 - 2%, while for $R_1/R_2 \geqslant 0.7$ it is fractions of a percent. The intensity factors $N_\pm/N_0$ for the annular sector are presented in the table.

5°. "Banana-shaped" crack. The line $\mathbf{R}(l)$ has the form of a semicircle of radius R and $\rho(l) = \sqrt{\cos\alpha}$, $\alpha = l/R$ (Fig. 6).

We have

$$Q = 1 - {}^1/_4\varepsilon\sqrt{\cos\alpha}m - \frac{\varepsilon^2}{32}\left[3\cos\alpha\ln\left(\frac{256}{\varepsilon^2}\operatorname{tg}\frac{\alpha_1}{2}\operatorname{tg}\frac{\alpha_2}{2}\Big/\cos\alpha\right) - 5\cos\alpha + 4\sin\alpha\operatorname{tg}\alpha - 6m^2\cos\alpha + \cos\alpha(\cos\alpha_1\operatorname{ctg}\alpha_1 + \cos\alpha_2\operatorname{ctg}\alpha_2) + 4\sin\alpha(\sin^{-1}\alpha_2 - \sin^{-1}\alpha_1)\right]$$

$$\alpha_1 = \frac{\pi - 2\alpha}{4},\quad \alpha_2 = \frac{\pi + 2\alpha}{4}$$

The intensity factor $N_\pm$ is in the table for $\alpha = 0$ (on the axis of symmetry).

6°. A constant width crack stretched along the arc of a parabola. The line $\mathbf{R}(l)$ is given parametrically: $x = a\alpha$, $y = a\alpha^2/2$, $|\alpha| \leqslant \alpha_0$, $\alpha_0 > 0$ (a has the meaning of the focal parameter of the parabola): $\rho(l) = a =$ const. For $l = \alpha = 0$ (at the parabola vertex) $\chi = 1$ and

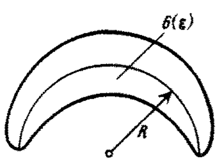


Fig. 6

$$Q(0, m, \varepsilon) = 1 - {}^1/_4\varepsilon m + \frac{\varepsilon^2}{16}\ln\frac{4}{\varepsilon} + \frac{9}{32}\varepsilon^2 + \frac{\ln 2}{8}\varepsilon^2 + \frac{3}{16}\varepsilon^2 m^2 - \varepsilon^2\left[\frac{\alpha_1^3}{3\alpha_0^3\alpha_2} + \frac{\alpha_1\alpha_2}{24\alpha_0^2} + \frac{1}{16}\ln\frac{2\alpha_1 + \alpha_2}{\alpha_0}\right]$$

$$\alpha_1 = \sqrt{1 + \alpha_0^2},\quad \alpha_2 = \sqrt{4 + \alpha_0^2}$$

The case $\alpha_0 \to \infty$ corresponds to an infinite crack of constant width stretched along a parabola. The quantity $Q(0, m, \varepsilon)$ here differs from the value of (3.1) by an amount $({}^1/_{16})\varepsilon^2 \ln 3$, i.e., the values of the intensity factor $N_\pm$ are not much less than for an annular crack with the same local characteristics (width and curvature).

REFERENCES

1. PANASIUK V.V., Ultimate equilibrium of brittle bodies with cracks, Naukova Dumka, Kiev, 1968.
2. GOL'DSHTEIN R.V., Plane crack of an arbitrary discontinuity in an elastic medium, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.3, 1979.

3. GOL'DSHTEIN R.V. and ENTOV V.M., Variational estimates for stress intensity factors on the contour of a plane crack of a normal discontinuity, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.3, 1979.
4. VLADIMIROV V.S., Generalized functions in mathematical physics, Nauka, Moscow, 1976.
5. RICE J., Mathematical methods in fracture mechanics /Russian translation/, Fracture, Vol. 2, Mir, Moscow, 1975.
6. MAZ'YA V.G., NAZAROV S.A., PLAMENEVSKII B.A., Asymptotic form of the solutions of elliptical boundary value problems for singular domain perturbations. Izdat. Tbilis. Univ., Tbilisi, 1981.
7. KALKER J.J., The surface displacement of an elastic half-space loaded in a slender bounded, curved surface region with application to the calculation of the contact pressure under the roller, J. Inst. Math. Applic., Vol.19, No.2, 1977.
8. JOHNSON R.E., An improved slender-body theory for Stokes Flow, J. Fluid Mech. Vol. 99, Pt. 2, 1980.
9. COLE J., Perturbation methods in applied mathematics /Russian translation/, Mir, Moscow, 1972.
10. NAYFEH A.H., Perturbation methods /Russian translation/, Mir, Moscow, 1976.
11. SMETANIN B.I., Problem of tension on an elastic space containing a plane annular slot, PMM, Vol.32, No.3, 1968.
12. MASTROJANNIS E.N. and KERMANIDIS T.B., An approximate solution of the annular crack problem, Intern. J. Numer. Meth. Engng., Vol.17, No.11, 1981.

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STATIONARY MOTIONS OF A GYROSTAT WITH AN ELASTIC ANNULAR PLATE AND THEIR STABILITY*

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Using Rumyantsev methods /1-3/ in the Kuz'min form /4/, stationary motions are deduced for a gyrostat with a circular annular plate clamped by the inner contour in a housing, and sufficient conditions are obtained for their stability. The paper touches on a cycle of papers devoted to investigating the stability of systems with distributed parameters: elastic rods, flexible rectangular plates, and a flexible string /5-19/.

1. We introduce the following coordinate system: $Cx_1x_2x_3$ is the orbital system with origin at the centre of mass of the mechanical system for the plate state of strain, the Cx_2 axis is along the orbit radius, the Cx_3 axis is perpendicular to the orbit plane, and the axis Cx_1 is orthogonal to the Cx_2, Cx_3 axes; $Oxyz$ is the coordinate system coupled rigidly to the gyrostat housing whose axes are directed along the principal central axes constructed for the centre of mass O of the system for the undeformed state of the plate; $Cy_1y_2y_3$ is the coordinate system whose y_s axes $(s=1,2,3)$ are parallel to the x, y, z axes, respectively.

We will define the gyrostat location in the orbital coordinate system by the Euler angles ψ, θ, φ and the direction of the x_s axes $(s=1,2,3)$ with respect to the axes of the system $Cy_1y_2y_3$ by the direction cosines $\alpha_{s1}, \alpha_{s2}, \alpha_{s3}$ that depend in a known manner on the angles ψ, θ, φ, for instance, $\alpha_{31} = \sin\varphi \sin\theta$ [20].

We will define the location of points of the plate in the deformed state with respect to the gyrostat housing by a radius-vector whose projections on the axes are

$$r_x = (a+r)\cos\lambda - zu_1, \quad r_y = (a+r)\sin\lambda - zu_2 \tag{1.1}$$
$$r_z = z + w \quad (u_1 = w_r\cos\lambda - (a+r)^{-1} w_\lambda \sin\lambda, \quad u_2 = w_r \sin\lambda + (a+r)^{-1} w_\lambda \cos\lambda)$$

Here a is the radius of the inner circular contour of the middle plane located in the Oxy plane, $a+r, \lambda, z$ are cylindrical coordinates of an arbitrary point of the plate in the undeformed state, $w(r, \lambda, t)$ is the projection of the elastic displacement vector of an arbitrary point of the middle plane on the z-axis, and the letter subscripts on the quantity w denote first-order partial derivatives with respect to the variable indicated in the subscript.

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